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Spectrum of Scalar Curvature Perturbations in Krein Space Quantization

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Abstract The main goal of this paper is to derive the primordial power spectrum for the scalar perturbations generated as a result of quantum fluctuations during an inflationary period by an alternative approach of field quantization, *i.e.* Krein space quantization (Gazeau et al. in Class. Quantum Gravity 17:1415, 2000; Takook in Int. J. Mod. Phys. 11:509, 2002; Rouhani and Takook in Int. J. Theor. Phys. 48:2740, 2009). The spectrum of scalar curvature perturbations are calculated in the slow roll approximation.

Keywords Power spectrum · Krein space quantization · Scalar curvature perturbations · Slow roll approximation

1 Introduction

In recent years it has been realized that much can be learnt about the highest energies and the smallest scales by studying cosmology and in particular the very early universe. An especially intriguing idea in this context is inflation. Inflation successfully solves several problems of the standard big bang scenario, and also makes a number of new predictions. The particular interest is the CMBR anisotropy, which currently is measured with higher precision.

Typically inflation is discussed from a purely field theoretic perspective, where the vacuum energy generates inflation. In the standard inflationary scenario the quantum fluctuations start out with a linear size much smaller than the Planck scale. Nevertheless it is assumed that no new physics appear, and a natural vacuum for the quantum fluctuations is

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chosen with this in mind. In this paper we have considered the combination of quantum field theory in Krein space together with consideration of quantum metric fluctuations, for derive formula for the spectrum of scalar curvature perturbations produced during inflation. The standard results to first order in the slow roll approximation in Hilbert space quantization are

$$P_{\Re}(k) = \left(\frac{H}{\dot{\phi}}\right)^2 P_\phi(k), \tag{1}$$

where $P_\phi(k) = (\frac{H}{2\pi})^2$ is spectrum of scalar field perturbation [11, 17]. This paper is devoted to calculation of power spectrum in Krein space quantization in the slow roll approximation.

Let us briefly describe our quantization of the minimally coupled massless scalar field in de Sitter space. It is defined by

$$\Box_H \phi(x) = 0,$$

where $\Box_H$ is the Laplace-Beltrami operator on de Sitter space. As proved by Allen [2], the covariant canonical quantization procedure with positive norm states fails in this case. The Allen's result can be reformulated in the following way: the Hilbert space generated by a complete set of modes (named here the positive modes, including the zero mode) is not de Sitter invariant,

$$\mathcal{H} = \left\{\sum_{k\geq 0} \alpha_k \phi_k;\ \sum_{k\geq 0} |\alpha_k|^2 < \infty\right\}.$$

This means that it is not closed under the action of the de Sitter group. Nevertheless, one can obtain a fully covariant quantum field by adopting a new construction [5, 6]. In order to obtain a fully covariant quantum field, we add all the conjugate modes to the previous ones. Consequently, we have to deal with an orthogonal sum of a positive and negative inner product space, which is closed under an indecomposable representation of the de Sitter group. The negative values of the inner product are precisely produced by the conjugate modes: $\langle \phi_k^*, \phi_k^* \rangle = -1$, $k \geq 0$. We do insist on the fact that the space of solution should contain the unphysical states with negative norm. Now, the decomposition of the field operator into positive and negative norm parts reads

$$\phi(x) = \frac{1}{\sqrt{2}}[\phi_p(x) + \phi_n(x)], \tag{2}$$

where

$$\phi_p(x) = \sum_{k\geq 0} a_k \phi_k(x) + H.C., \quad \phi_n(x) = \sum_{k\geq 0} b_k \phi^*(x) + H.C.. \tag{3}$$

The positive mode $\phi_p(x)$ is the scalar field as was used by Allen. The crucial departure from the standard QFT based on CCR lies in the following requirement on commutation relations:

$$a_k|0\rangle = 0, \quad [a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad b_k|0\rangle = 0, \quad [b_k, b_{k'}^\dagger] = -\delta_{kk'}. \tag{4}$$

A direct consequence of these formulas is the positivity of the energy *i.e.*

$$\langle \vec{k}|T_{00}|\vec{k}\rangle \geq 0,$$

for any physical state $|\vec{k}\rangle$ (those built from repeated action of the $a_k^\dagger$'s on the vacuum). This quantity vanishes if and only if $|\vec{k}\rangle = |0\rangle$. Therefore the "normal ordering" procedure for eliminating the ultraviolet divergence in the vacuum energy, which appears in the usual QFT is not needed [6]. Another consequence of this formula is a covariant two-point function, which is free of any infrared divergence [18]:

$$\mathcal{W}(x,x') = \frac{iH^2}{8\pi}\epsilon(x^0 - x'^0)[\delta(1-\mathcal{Z}(x,x')) - \theta(\mathcal{Z}(x,x')-1)],$$

where

$$\epsilon(x^0 - x'^0) = \begin{cases} 1 & x^0 > x'^0 \\ 0 & x^0 = x'^0 \\ -1 & x^0 < x'^0 \end{cases},$$

and θ is the Heaviside step function and $\mathcal{Z}(x,x') = \cosh H\sigma$. σ is the invariant geodesic distance between the two points in de Sitter space and H is the Hubble parameter.

2 Power Spectrum of the Scalar Curvature Perturbations

The metric in 4 space-time dimensions chosen as:

$$ds^2 = dt^2 - a^2(t)(\overrightarrow{dx}.\overrightarrow{dx}) = a^2(\eta)[d\eta^2 - (\overrightarrow{dx}.\overrightarrow{dx})]. \tag{5}$$

Scalar linear perturbations to this metric can be expressed most generally as [3, 10]

$$ds^2 = a^2(\eta)\{(1+2A)d\eta^2 - 2\partial_i B dx^i d\eta - [(1+2D)\delta_{ij} + 2\partial_i\partial_j E]dx^i dx^j\}, \tag{6}$$

where $i, j = 1, 2, 3$. The scalar perturbation amplitudes A, B, D, E are functions of space and time coordinates. On the other hand, the intrinsic curvature perturbation of co-moving hyper-surfaces is denoted by $\Re$, and during inflation, is given by [8]

$$\Re = D - \frac{H}{\dot{\phi}}\delta\phi, \tag{7}$$

where $\delta\phi$ is the perturbation in the inflaton field. Its spectrum is defined by

$$\Re = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Re_{\mathbf{k}}(\eta)e^{i\mathbf{k}.\mathbf{x}}, \tag{8}$$

$$\langle 0|\Re_{\mathbf{k}}\Re^*_{\mathbf{l}}|0\rangle = \frac{2\pi^2}{k^3}P_{\Re}\delta^3(\mathbf{k}-\mathbf{l}). \tag{9}$$

Units are such that $c = \hbar = 8\pi G = 1$. ϕ is the inflaton field and a dot denotes the derivative with respect to time t.

3 Power Spectrum in Krein Space Quantization

In the previous works, we have shown that presence of negative norm states *i.e.* Krein space quantization, play the role of an automatic renormalization device for certain problems [6, 9, 15, 20, 21]. The negative modes do not interact with the physical states or real physical world, thus they can not be affected by the physical interaction as well. On the other hand, the auxiliary negative norm states in our problem are similar to the ghost states in the standard gauge QFT. In the gauge QFT, the auxiliary negative norm states (ghost states), can neither propagate in the physical world nor interact with physical states. In physics of the very early universe, probably these states may interact with physical states, and therefore unitarity may not be defined. But in the present world, the particles inside horizon do not affect over these states, and hence unitarity is restored at the low energies that now occur in the universe [7].

In the Minkowski space, the ultraviolet divergence of the vacuum energy of the quantum field is removed by the normal ordering. In the curved space-time, the standard renormalization of the ultraviolet divergence of the vacuum energy is accomplished by subtracting the local divergencies of Minkowski space [4],

$$\langle \Omega \mid : T_{\mu\nu} : \mid \Omega \rangle = \langle \Omega \mid T_{\mu\nu} \mid \Omega \rangle - \langle 0 \mid T_{\mu\nu} \mid 0 \rangle,$$

where $\mid \Omega\rangle$ is the vacuum state in curved space and $\mid 0\rangle$ is the vacuum state in Minkowski space. The mines sign in the above equation can be interpreted as the negative norm states which is added to the field operator. This interpretation resembles Krein space quantization where the negative norm states are considered as well. These negative norm states however, are defined in Minkowski space alone and they are not the solution of the wave equation in the curved space time. In other words this renormalization vividly breaks the symmetry of the curved space.

There are other possibilities for removing the local divergencies in the curved space, the so called non-standard renormalization schema. The local divergencies of curved space is removed by the quantities defined in the same curved space-time. In this case, the mines sign can be interpreted as the negative norm states which is added to the field operator and they are the solutions of the wave equation in the curved space-time. This scheme seams to be more logical than to standard one, since the curved space symmetry has been preserved after renormalization procedure.

The highest energies regime of early universe is only exclusive positions for appearance of maximum symmetry between all interactions and realization of quantum gravity. Quantum gravity theory is required in order to understand problems involving the combination of very high energy and very small dimensions of space, such as the behavior of the origin of the universe. In order to have a correct quantum gravity, one of the possibility is the four or higher derivatives gravity. However, these higher derivatives seem to lead to the ghosts states, states with negative norm [7]. Departures from unitarity for higher derivative gravity is a interesting problem for some authors [12].

Now we calculate the power spectrum of scalar curvature perturbations in Krein space quantization. The effective action during inflation is assumed to be [3]

$$S = -\frac{1}{2}\int R\sqrt{-g}d^4x + \int\left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right]\sqrt{-g}d^4x. \tag{10}$$

The action for scalar linear perturbations is [17]

$$S = \frac{1}{2}\int\left[(u')^2 - (\partial_i u)^2 + \frac{z''}{z}u^2\right]d\eta d^3\mathbf{x}, \tag{11}$$

where $z = \frac{a\dot{\phi}}{H}$, prime denotes the derivative with respect to conformal time η and during inflation [14, 17]

$$u = -z\Re. \tag{12}$$

The effective Lagrange for u is

$$\$ = \frac{1}{2}\left[(u')^2 - (\partial_i u)^2 + \frac{z''}{z}u^2\right]. \tag{13}$$

This is just a free theory with a time-dependent mass $m^2 = -\frac{z''}{z}$. Therefore the quantization is straightforward.

To quantize the theory, we use the field and its conjugate momentum in the momentum picture

$$u_{\mathbf{k}}(\eta) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} u(\eta, \mathbf{x}) e^{-i\mathbf{k}.\mathbf{x}}, \tag{14}$$

$$\Pi_{\mathbf{k}}(\eta) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} \Pi(\eta, \mathbf{x}) e^{-i\mathbf{k}.\mathbf{x}}. \tag{15}$$

Note that $u^\dagger_{\mathbf{k}} = u_{-\mathbf{k}}$, $\Pi^\dagger_{\mathbf{k}} = \Pi_{-\mathbf{k}}$. From (13) we have $\Pi = \frac{\partial \$}{\partial u'} = u'$, which using (14) and (15) translates to $u'_{\mathbf{k}}(\eta) = \Pi_{\mathbf{k}}(\eta)$ for the Fourier transforms. Also the canonical commutation relations are

$$[u(\eta, \mathbf{x}), u(\eta, \mathbf{x}')] = [\Pi(\eta, \mathbf{x}), \Pi(\eta, \mathbf{x}')] = 0, \tag{16}$$

$$[u(\eta, \mathbf{x}), \Pi(\eta, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}'). \tag{17}$$

Also one can check that the canonical commutation relations (17) imply $[u_{\mathbf{k}}(\eta), \Pi^\dagger_{\mathbf{q}}(\eta)] = i\delta^3(\mathbf{k} - \mathbf{q})$. The Fourier modes of u obey the field equation [3]

$$u''_{\mathbf{k}} + \left(k^2 - \frac{z''}{z}\right)u_{\mathbf{k}} = 0, \tag{18}$$

where $k = |\mathbf{k}|$. The time-independent normalization is chosen to be

$$u^*_{\mathbf{k}} u'_{\mathbf{k}} - u_{\mathbf{k}} u'^*_{\mathbf{k}} = -i. \tag{19}$$

The mode expansion of the field operator $u(\eta, \mathbf{x})$ is

$$\begin{aligned} u(\eta, \mathbf{x}) = (2\pi)^{-3/2} \int d^3\mathbf{k}\{&[u_{k,p}(\eta)a(\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}} + u^*_{k,p}(\eta)a^\dagger(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}}] \\ &+ [u_{k,n}(\eta)b(\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}} + u^*_{k,n}(\eta)b^\dagger(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}}]\}, \end{aligned} \tag{20}$$

where $u_{k,p}$ and $u_{k,n}$ are modes related to positive and negative norm states respectively. The operators $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$, $b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ are defined in [13]. At very short wave length, $\frac{k}{aH} \gg 1$, solutions are

$$u_k(\eta) \sim \frac{1}{\sqrt{2k}} e^{\pm ik\eta}. \tag{21}$$

In the opposite long-wave regime, $\frac{k}{aH} \ll 1$, where k can be neglected in (18), we see that the growing mode solution is

$$u_k \sim z, \tag{22}$$

i.e. $\frac{u_k}{z}$ and thus $\Re$ is constant on super-horizon scales. One has from [17]:

$$\frac{z''}{z} = 2a^2H^2\left(1 + \frac{3}{2}\delta + \epsilon + \frac{1}{2}\delta^2 + \frac{1}{2}\epsilon\delta + \frac{1}{2}\frac{1}{H}\dot{\epsilon} + \frac{1}{2}\frac{1}{H}\dot{\delta}\right), \tag{23}$$

where

$$\epsilon \equiv \frac{-\dot{H}}{H^2}, \quad \delta \equiv \frac{\ddot{\phi}}{H\dot{\phi}}, \tag{24}$$

and

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a^2H} = \frac{-1}{aH} + \int \frac{\epsilon da}{a^2H}. \tag{25}$$

Thus if ϵ and δ are constant, one obtain

$$\eta = \frac{-1}{aH}\frac{1}{1-\epsilon}, \quad (N.B.{:}\epsilon < 1 \leftrightarrow \text{inflation}), \tag{26}$$

$$\frac{z''}{z} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right), \tag{27}$$

where

$$\nu = \frac{1-\delta+\epsilon}{1-\epsilon} + \frac{1}{2}. \tag{28}$$

Therefore we have from (8) and (12) and (20)

$$\langle 0|\Re_{\mathbf{k}}\Re^*_{\mathbf{l}}|0\rangle = \left(\frac{|u_{k,p}(\eta)|^2}{z^2} - \frac{|u_{k,n}(\eta)|^2}{z^2}\right)\delta^3(\mathbf{k}-\mathbf{l}). \tag{29}$$

From (9) and (29), power spectrum is

$$P_{\Re}(k) = \frac{k^3}{2\pi^2}\left(\frac{|u_{k,p}(\eta)|^2}{z^2} - \frac{|u_{k,n}(\eta)|^2}{z^2}\right). \tag{30}$$

4 Power Spectrum in Slow-Roll Approximation

The orthogonal modes u_k, u_k^* of (18) are easy to construct in the slow roll approximation, when ϵ and δ are small, $(\epsilon, \delta \ll 1)$. During slow roll inflation, to $O(\epsilon, \delta)$, we have $(1-\epsilon)\eta = -\frac{1}{aH}$ and so $\frac{z''}{z} = \frac{2-3\delta+6\epsilon}{\eta^2}$. Then the mode equation (18) become [8, 17]

$$u_k'' + \left(k^2 - \frac{2-3\eta+6\epsilon}{\eta^2}\right)u_k = 0. \tag{31}$$

The standard choice of the modes u_k, u_k^* is to take

$$u_k(\eta) = -\frac{\sqrt{\pi\eta}}{2}H_\nu^{(-)}(k\eta), \quad u_k^*(\eta) = -\frac{\sqrt{\pi\eta}}{2}H_\nu^{(+)}(k\eta), \tag{32}$$

as the positive and negative frequency modes, respectively, where $\nu = \frac{3}{2} - \delta + 2\epsilon$ [17]. From the asymptotic formula for large $k\eta$ [1]

$$H_\nu^{(-)}(k\eta) \sim \sqrt{\frac{2}{\pi k\eta}}\left[1 - i\frac{4\nu^2 - 1}{8k\eta}\right]\exp\left[-ik\eta - i\pi\left(\frac{\nu}{2} + \frac{1}{4}\right)\right]. \tag{33}$$

So

$$u_k(\eta) = \frac{1}{\sqrt{2k}}\left[1 - i\frac{4\nu^2 - 1}{8k\eta}\right]\exp\left[-ik\eta - i\pi\left(\frac{\nu}{2} + \frac{1}{4}\right)\right]. \tag{34}$$

And complex conjugate of (34) will be mode corresponding of the negative frequency. Ignoring the slow roll corrections, in which case the modes (34) reduce to

$$u_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}, \quad u_k^* = \frac{1}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right)e^{ik\eta}. \tag{35}$$

Accordingly to the second perspective of [13] and using (34), to leading order of slow roll parameter, we have

$$u_{k,p} = \frac{1}{\sqrt{2k}}\left(1 - i\frac{1 + 3\epsilon - \frac{3}{2}\delta}{k\eta}\right)e^{-ik\eta}, \quad u_{k,n} = \frac{1}{\sqrt{2k}}\left(1 + i\frac{1 + 3\epsilon - \frac{3}{2}\delta}{k\eta}\right)e^{ik\eta}. \tag{36}$$

Then

$$\langle 0|\Re_{\mathbf{k}}\Re_{\mathbf{l}}^*|0\rangle = \frac{2\pi^2}{k^3}\left(\frac{H}{\dot\phi}\right)^2\left(\frac{H}{2\pi}\right)^2\left[\frac{k}{H}e^{-\alpha k^2} + 6\epsilon - 3\delta\right]\delta^3(\mathbf{k} - \mathbf{l}), \tag{37}$$

where $\alpha = \frac{1}{\pi H^2}$. Therefore we have for power spectrum curvature

$$P_\Re(k) = \left(\frac{H}{\dot\phi}\right)^2\left(\frac{H}{2\pi}\right)^2\left[\frac{k}{H}e^{-\alpha k^2} + 6\epsilon - 3\delta\right]. \tag{38}$$

On the other hand, accordingly to the first perspective of [13], using the above physical mode of (36) and also (14) of [13] as the non-physical mode, we have

$$u_{k,p} = \frac{1}{\sqrt{2k}}\left(1 - i\frac{1 + 3\epsilon - \frac{3}{2}\delta}{k\eta}\right)e^{-ik\eta}, \quad u_{k,n} = \frac{1}{\sqrt{2k}}e^{ik\eta}, \tag{39}$$

then

$$\langle 0|\Re_{\mathbf{k}}\Re_{\mathbf{l}}^*|0\rangle = \frac{2\pi^2}{k^3}\left(\frac{H}{\dot\phi}\right)^2\left(\frac{H}{2\pi}\right)^2(1 + 6\epsilon - 3\delta)\delta^3(\mathbf{k} - \mathbf{l}). \tag{40}$$

Therefore perturbation's power spectrum to leading order of slow roll parameter will be:

$$P_\Re(k) = \left(\frac{H}{\dot\phi}\right)^2\left(\frac{H}{2\pi}\right)^2(1 + 6\epsilon - 3\delta), \tag{41}$$

we see with ignoring slow roll corrections, our answer reduced to standard result (1).

5 Conclusion

The negative frequency solutions of the field equations are needed for the covariant quantization in the minimally coupled scalar field in de Sitter space. Contrary to the Minkowski space, the elimination of de Sitter negative norms in this case breaks the de Sitter invariance. In other words, in order to restore the de Sitter invariance, one needs to take into account the negative norm states *i.e.* the Krein space quantization. The spectrum of scalar curvature perturbations and gravitational waves has been calculated through the Krein space quantization exhibiting. Once again the theory is automatically renormalized. This is means that the presence of negative norm states in the structure of very early universe preserve de Sitter invariance and automatically removes singularity of the theory.

Moreover, for new spectrum the scale invariance is broken. This is similar to the works of Kaloper et al. where the scale invariance is broken due to consideration of perturbations of the gravitational field [8]. These perturbations have the trace of quantum gravity in the very early universe with the curved background.

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